Math 304 (Spring 2015) - Homework 9

Problem 1.

Find the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

(a) First solve

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)-8 = \lambda^2-4\lambda-5 = 0$$

we get

$$\lambda = 5, -1$$

When $\lambda = 5$, compute the kernel

$$A - 5I = \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}$$

we find that $v = (1,1)^T$ is an eigenvector belonging to $\lambda = 5$. Similarly, when $\lambda = -1$, we find $w = (1,-2)^T$ is an eigenvector belonging to $\lambda = -1$.

(b) Use the method as in part (a). I skip the details.

Eigenvalues: $\lambda = 3+4i, 3-4i$.

$v = (2i, 1)^T$ is an eigenvector belonging to $\lambda = 3+4i$.

$w = (-2i, 1)^T$ is an eigenvector belonging to $\lambda = 3-4i$.

(c) Eigenvalues: $\lambda = 2, 1$.

Eigenvectors:

$v = (1,1,0)^T$ is an eigenvector belonging to $\lambda = 2$.

$w_1 = (1,0,0)^T$ and $w_2 = (0,1,-1)^T$ are two linearly independent eigenvectors belonging to $\lambda = 1$.

Problem 2.

Determine whether the matrix $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ is diagonalizable.

Solution: Compute the eigenvalues of A. We find only one eigenvalue $\lambda = 1$.

Thus we need to further determine whether there are two linearly independent eigenvectors or not.

Solve for eigenvectors, and we find only one linearly independent eigenvector $v = (1, 0)^T$. Therefore, A is not diagonalizable.

Problem 3.

In each of the following, write the matrix A as a product SDS^{-1}, where D is a diagonal matrix.

(a) $\begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

Solution:

(a) First, find the eigenvalues and corresponding eigenvectors.

Eigenvalues: $\lambda = 2, 1$.

Eigenvectors:

$v = (2, -1)^T$ is an eigenvector belonging to $\lambda = 2$.

$w = (3, -2)^T$ is an eigenvector belonging to $\lambda = 1$.

Let

$$S = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $A = SDS^{-1}$.

(b) Eigenvalues: $\lambda = \pm 1$.

Eigenvectors:

$v = (1, -1)^T$ is an eigenvector belonging to $\lambda = -1$.

$w = (1, 1)^T$ is an eigenvector belonging to $\lambda = 1$.

Let

$$S = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $A = SDS^{-1}$.

(c) Eigenvalues: $\lambda = 1, 2, -2$.

Eigenvectors:

$v = (-1, 1, 0)^T$ is an eigenvector belonging to $\lambda = 1$.

$w = (0, 3, 1)^T$ is an eigenvector belonging to $\lambda = 2$.

$u = (0, 1, -1)^T$ is an eigenvector belonging to $\lambda = -2$.

Let

$$S = \begin{pmatrix} | & | & | \\ v & w & u \\ | & | & | \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Then $A = SDS^{-1}$.

Problem 4.

Compute e^A of the matrix $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$.

Solution: Eigenvalues: $\lambda = \pm 1$.

Eigenvectors:

$v = (2, -1)^T$ is an eigenvector belonging to $\lambda = 1$.

$w = (1, -1)^T$ is an eigenvector belonging to $\lambda = -1$.

Let

$$S = \begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $A = SDS^{-1}$. It follows that

$$e^A = Se^DS^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 2e - e^{-1} & 2e - 2e^{-1} \\ -e + e^{-1} & -e + 2e^{-1} \end{pmatrix}$$

Problem 5.

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}$.

Solution: Eigenvalues: $\lambda = 0, 3$.

Eigenvectors:

$v = \begin{pmatrix} 1-i \\ -2 \end{pmatrix}$ is an eigenvector belonging to $\lambda = 0$.

$w = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$ is an eigenvector belonging to $\lambda = 3$.